1. The Geometrization of Geographical Space in the Paradigm of Centrality

Almost all geographers in our time accept two of the fundamental ideas on centrality attributed to Christaller :

(1) On the earth's surface, the more a place is occupied the more it is a concentration of commerce, services, and functions (Christaller even added of industries).

(2) The relationships among inhabited places depend not only on the correlation of population and functions, but also on the location of each place in relation to the others. Some places are "more central than others" and there is therefore a hierarchy of central places.
The following two ideas of Christaller are now less readily accepted.

(3) The geometry of central places conforms to a regular triangulo-hexagonal model. Six equilateral triangles having a common summit are, indeed, visually associated with a regular hexagon.

(4) If a central place is put by hypothesis in the center of a regular hexagon, geometrical arrangements of central places are obtained according to where the others are put that conform to different principles. (4.1.) If the other places are arranged at the summits of the hexagon, they operate according to a "marketing principle" (k = 3). (4.2.) Arranged at the middle of the sides, they enter into a "transit" or "traffic principle" (k = 4). (4.3.) Arranged on a regular hexagon inside the initial regular hexagon, they conform to a "community and governmental principle" (k = 7).

These things being recalled, since no mathematical demonstration of the geometry of centrality is given by Christaller, two interpretations of it were proposed in 1965 by Anglo-Saxon geographers. The first (Michael F. Dacey : "The Geometry of Central Place Theory", Geografiska annaler, series B, Human geography, 1965, vol. 47, N° 2, pp. 111-124) emphasizes the lattice formed by central places. The second (Peter Haggett : "Locational Analysis in Human Geography", Arnold, 1965) uses the theory of covering a surface by means of a single regular geometrical form (the territory concept). These two interpretations are nonetheless based on a single mathematical concept, which can be summarized in the following way. Supposing that through two points there can pass only one straight line and that through a point not on that line there can be drawn only one parallel to it, it is possible to show how, in a system of axes drawn from a point of origin, a segment of a straight line with a direction (a vector) can be displaced in a manner determined by translation and rotation. Finally, supposing that on the plane thus defined the shortest distance between two points will be a straight line segment, we have produced a Euclidean plane with its well-known properties (reciprocity of distances, additivity of lengths, inequality of the triangle).

In his 1965 paper, Dacey draws six axes from a point of origin on a Euclidean plane at 60° ($\Pi/3$) intervals. In this way he obtains "a place symmetry lattice with a six fold or hexagonal axis" (p. 111). Next he shows that the translation and rotation of a vector placed at the origin of the axial system can produce an equilateral triangle. Finally, rotation of the equilateral triangle allows the construction of an isolated regular hexagon that is contiguous with or included in other regular hexagons. Dacey calls the equilateral triangle used to "generate" all the hexagonal figures a "unit cell".

It is intuitively obvious that other systems of axes (having angles of any size between them) can be placed at the point of origin of the plane. From this it is clear, as Dacey points out, that the choice of a system of axes having angles of $\Pi/3$ is a matter of convention or of convenience. There are advantages, for example, to the simple numerical expressions of Christaller (k = 3, k = 4, k = 7) which can be generalized (k = 9, k = 12, k = 13, etc.) and used to compute the number of localized functions in the central places of a regular hexagonal lattice. The same year (1965), Peter Haggett pointed out that, measured by the distance of displacement between the center and the summits, the efficiency of movement is nearly optimal in a regular hexagon (only the circle has a better efficiency, if taken by itself). Measured by the ratio of the perimeter to the area of the figure, the regular hexagon is again, except for the circle, the most satisfactory geometrical shape (efficiency of boundaries). However, in contradistinction to a circle, an area filled with contiguous regular of the same radius includes no voids among its unit cells. The covering of a Euclidean plane with regular hexagons is therefore the most "economical" because it combines two kinds of nearly optimal efficiency with continuous occupation of the area.

This being the case, and regardless of the attitude adopted thereafter as to the spatial models of Christaller, Lösch, and Isard, if we accept the regular or irregular triangulo-hexagonal model as the primary and normative means of representing centrality, we must also accept the following epistemological relationships between geometrical and geographical space. Namely, we must accept the hypothesis that the space of centrality is a homogeneous plane where all resources are arranged uniformly and all means of transport are initially in uniform distribution (transport surface) (Dacey, p. 112; Haggett, p. 135). This property follows from the fact that the mathematical interpretation of the theory proposed by Dacey and Haggett is that of a Euclidean plane. The property is based implicitly on the impossibility of reducing any empirical graphic representation of a lattice of geometrically irregular central places to its regular triangulo-hexagonal model. In other words, any graphic representation of a central geographical space is a holistic empirical distortion of the explanatory regular mathematical model. Hagett writes that "hexagons may ... be thought to be latent in most human organization but only through appropriate transformations (sic) of geographical space is their form likely to be made visible" (p. 55). Unfortunately this "ideal" model makes it necessary to distort reality to the point of rendering it unrecognizable and still does not resolve the geometrical problem of centrality in the way that Christaller stated it, since Dacey and Haggett reversed the relationship between geographical and (Euclidean) geometrical space.

2. Formulation of the Geometrical Problem of Centrality by Christaller

In what follows (sections 2 and 3), sentences or paragraphs beginning with "it is demonstrated that" refer to mathematical demonstrations undertaken and verifiable in the french text entitled "Le cadavre exquis de la centralité : l'adieu à l'hexagone régulier", p.49-68. According to Christaller, centrality is a principle (die zentralistische Prinzip) that is found concurrently in the human mind, in society and in nature. Centralization is a principle of order (Zentral als Ordnungsprinzip) founded on the relation between a center and its surroundings. A center is characterized by "the arrangement of mass around a nucleus" (die Anordnung einer Masse um einen Kern, ein Zentrum "p. 21 of the original German; "the crystallization (sic) of mass around a nucleus is ... a centralistic order" p. 14 of the English translation). Observation of inhabited places on the surface of the earth shows that, in medieval towns, there are structures the presence and design of which bear witness to their central character : church, town hall, town square, school, etc. In our times, even if it is no longer obvious, this order appears in the fact that each region is complementary (Ergänzungsgebiet) to a center where a certain number of "central functions" are located.

Central functions (die zentralen Funktionen) fall into nine categories :

(1) administrative
(2) religious and cultural
(3) public health
(4) mass communications
(5) cooperatives
(6) financial and commercial
(7) professional
(8) employment
(9) transportation and communication

Within a single category the functions are ordered hierarchically according to the extent and the frequency of the activity in a single central place.

A "central place" (das zentrale Ort; abbreviation : CP) is a "spatial organ" (ein räumliches Organ) in the origin of which "economic distance" plays a deciding but not exclusive role. The economic distance is a combination of the geographical distance (expressed in kilometers), the price of transportation (insurance and storage included) and various advantages of "transit" that are not all of the economic sort. Every CP serves a population the size of which is in proportion to the different types of functions and of goods used or supplied from the places where the inhabitants reside, whether they be dispersed or concentrated. For Christaller a "central good" (das zentrale Gut; abbreviation : CG) has a "range" (die Reichweite eines zentralen Gutes) which is the attainable "distance" (Entfernung) of a good supplied by a "central place" (ein zentrales Ort) to a dispersed or concentrated population. This "extent" is "absolute" (die ideale (absolute) Reichweite) when the place supplied is outside the "range" of any other CP for the good under consideration. It is "relative" (die relative Reichweite) if this good can be delivered more reasonably (price, proximity) from another center (German text, p. 65). This being assumed, Christaller considers every place outside the "range" (Reichweite) of a CG to be beyond the upper limit of the range of this CG. Similarly, the minimal quantity of the same CG that must be sold on the spot (to be profitable) at the CP determines the lower limit of the range of this CG. This would be the "smallest place" where population tends to gather. From this it is deduced that the relative range of a CG is between the absolute limits (upper and lower) of the CG. For, by definition, it must be possible to study whether the same CG can be distributed at a better price by other CP's in the area of the "complementary region" included in the "relative limit" (figure 1). Beyond the upper limit, all the other CP's distributing the same CG as the initial CP can distribute this CG at a profit. The conditions stated by Christaller as necessary to the solution of the geometrical problem of the CG's are therefore as follows : (1) every CG must be distributed (supplied) from the corresponding CP, (2) a CG is distributed (supplied) in a ring located around the corresponding CP, between the upper and lower limits of the "range" of that CG.

In other words, a CG_k of a "range" of K kilometers must be distributed only by the CP_k's, between K kilometers of the upper limit and 1 km of the lower limit. Christaller reasons as follows. Let there be a CG distributed from a CP called B. If the CG is distributed within a radius of 21 km around B, it is, by definition, the central good N° 21 (CG_{21}). Now let CG_{20} be distributed from B : CG_{20} will not be distributed in a ring situated between 20 and 21 km around B because it has an upper limit of 20 km. How, then, can the CG_{20} be distributed in the ring of 20-21 km ?

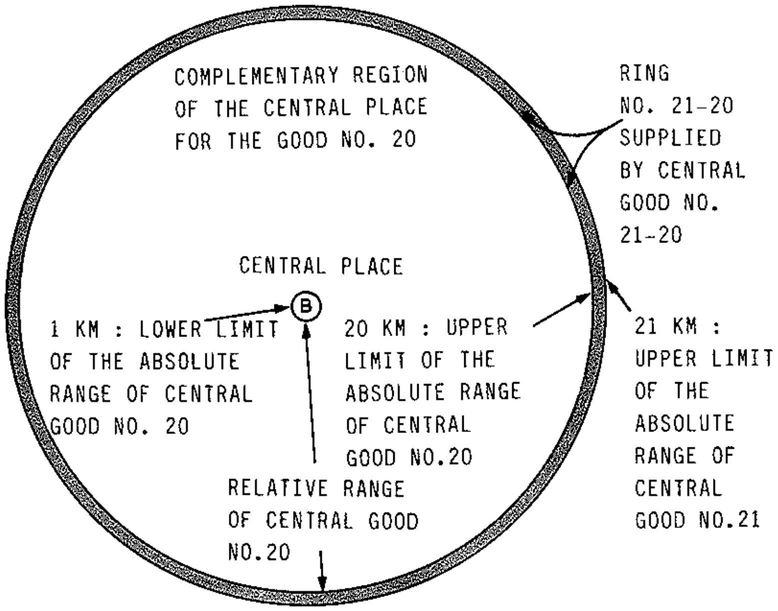

FIGURE 1 : THE PROBLEM OF DISTRIBUTION OF THE CENTRAL GOOD ACCORDING TO WALTER CHRISTALLER (1933)

(2.1.) Christaller states, without giving a demonstration, that if the "complementary region" of B is "isolated" at least three central places B 1, B 2, B 3 equidistant from each other are necessary to supply the ring N° 21-20 with the CG_{20}.

These three new CP's (each a $CPB_{k=20}$) can be located anywhere in the complementary region around B, except within a radius of 1 km around B, which is the "lower limit" of the good N° 20. Figure 1 (page 66 of the German edition) shows that these new CP's, each a B_{20}, are put by Christaller at 21 km from the initial central place B.

This choice cannot fail to be surprising in view of the concept of "relative range" of a good. For if it is obvious that the complementary regions located beyond 20 km are supplied without competition by other CPB_{20}'s, why limit the emplacement of these new CPB_{20}'s to the circumference of radius 21 alone when, just previously, Christaller asserted that they could be put anywhere outside the radius of 1 km around the initial $CPB_{20 \text{ and } 21}$?

Moreover, it is demonstrated and calculated that, geometrically, Christaller's assertion is false : three CPB_{20}'s, equidistant from each other and located in the ring N° 21-20 around the initial $CPB_{20 \text{ and } 21}$, are not sufficient to supply the whole ring N° 21-20 of an "isolated region", whether the CPB_{20} be located on the interior or on the exterior circle of the ring (figures 2 and 3).

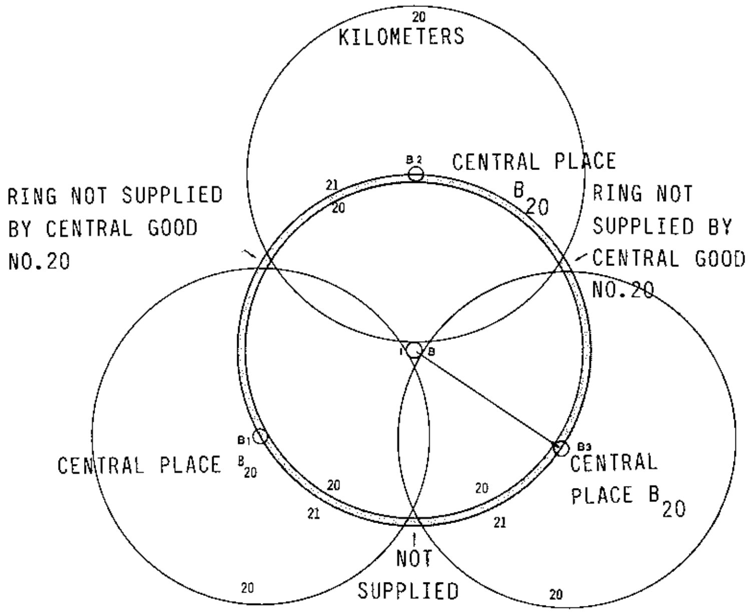

FIGURE 2 : HYPOTHESIS : ISOLATED COMPLEMENTARY REGION
THE CPB_{20} 'S : B1,B2,B3 LOCATED AT 21 KM
ON THE INITIAL $CPB_{20 \text{ AND } 21}$ BY CHRISTALLER
DO NOT SUPPLIED THE WHOLE RING NO.20-21

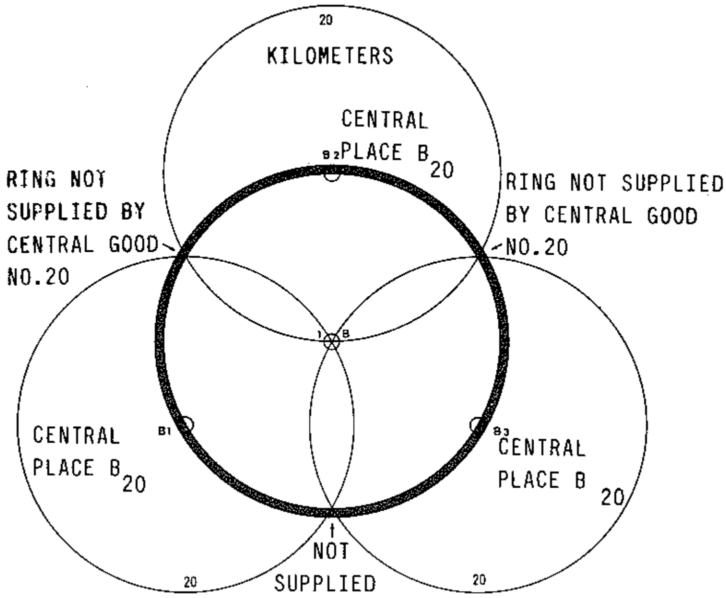

FIGURE 3 : HYPOTHESIS : ISOLATED COMPLEMENTARY REGION
THE CPB_{20}'S : B1,B2,B3 LOCATED AT 20 KM ON THE
INITIAL $CPB_{20 \text{ AND } 21}$ BY CHRISTALLER DO NOT SUPPLY
THE WHOLE RING NO.21-20 AND DO NOT OBSERVE THE
"LOWER LIMIT" OF CG_{20}

(2.2.) Christaller states, again without giving a demonstration, that if the "complementary region" of B is not isolated at least six other CG's aroud the initial CG are necessary to supply the ring N° 21-20 with the CG_{20}. First of all, the new CPB_{20} must be placed at 36 km around the initial $CPB_{20 \text{ and } 21}$. Then it will be possible to construct equilateral triangles the centers K of which are at 21 km from the initial $CPB_{20 \text{ and } 21}$ and the three summits of which are at 36 km from the initial $CPB_{20 \text{ and } 21}$ (figure 4). For if we assume in an equilateral triangle that

r = the side and h = the height, then $h = r \frac{\sqrt{3}}{2}$; but if

$r = BB\ 1 = 36$ km, then $BK = BB\ 1 \ \frac{\sqrt{3}}{2} \cdot \frac{2}{3} = 20.78 \simeq 21$ km.

Christaller then calculates a series of equilateral triangles such that center of gravity of the first becomes the summit of the next. Christaller then transforms the initial $CPB_{20 \text{ and } 21}$ into a CPG distributing the CG_{20}, CG_{21} and CG_{36}. On the other hand, the CPK and the CPB distribute only the CG_{20} and CG_{21}. The lattice is ordered gradually as it is constructed by means of the variations in "ranges" of the CG's. Finally, if the B's and K's are connected respectively among themselves, we obtain the famous regular hexagons. We see from figure 5, however, that ring N° 21-20 of the centers K is not entirely covered because a CPB is missing at each summit B of the hexagon with center B (which has become G). It is therefore necessary to add new CPB's constantly in order to supply the outer parts of the rings. This "solution" can be achieved only if the entire earth's surface is covered with CP's arranged on a regular hexagonal lattice of constant dimension. That this is an exorbitant pretension clashing with simple observation is shown by many empirical studies

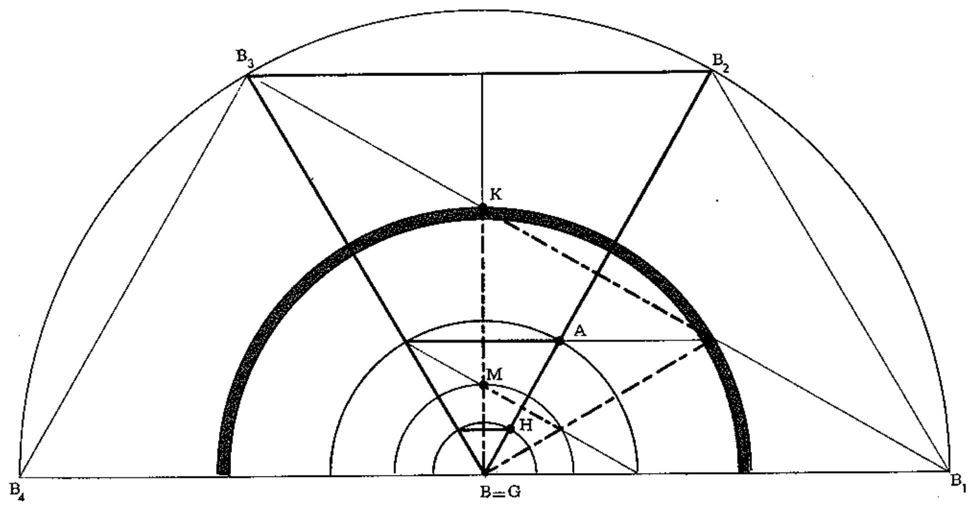

FIGURE 4

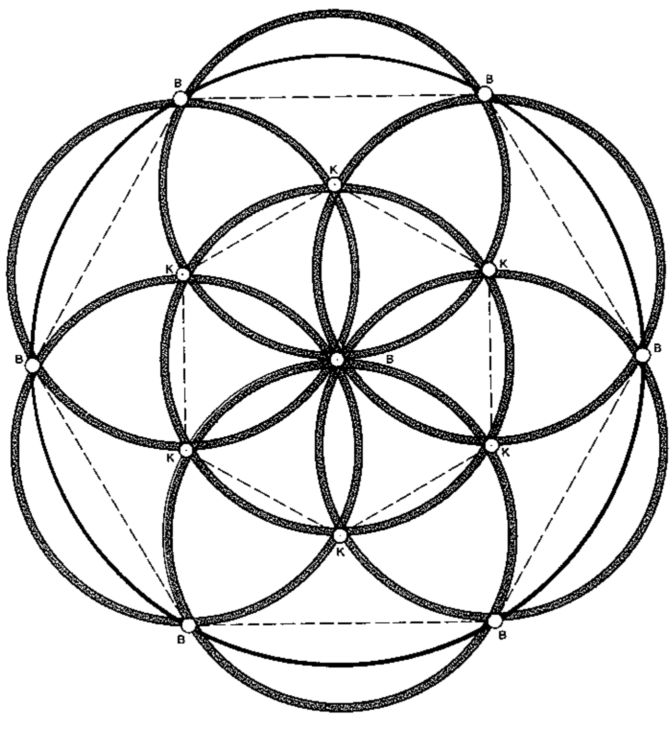

FIGURE 5 : COMPLEMENTARY REGION THAT IS NOT ISOLATED : CONSTRUCTION OF THE REGULAR HEXAGON AND HIERARCHICHAL ORDERING OF THE CENTRALIZED LATTICE ACCORDING TO CHRISTALLER (1933)

conducted around the world. These make it clear that the figure of the regular hexagon is not verified and that the dimensions of the observed figures (lozenges or pentagons or irregular hexagons, etc.) vary from one country to another. Finally, far from capable of being constructed on a "transport surface" of infinite dimension, the hexagonal system of central places of Christaller must be constructed in a finite space if it is not to remain unfinished. Consequently, supposing that the "transport surface" is a Euclidean plane (cf. Dacey, Haggett), the geometrical solution to centrality given by Christaller is either false, in case of the equilateral triangle, or inapplicable, in the case of the regular hexagon since in this second case the plane is infinite. Supposing, on the other hand, that the central space is finite (like the surface of the earth), the geometrical solutions of Christaller are never verified empirically. It remains to be determined whether, still using Christaller's formulation, it is possible to find a geometrical solution to the problem of centrality based on a primarily geographical and no longer exclusively geometrical conception of space.

3. Geometrical Solution to the Centrality Problem Stated by Christaller

Supposing, with Christaller, that there is a relation between the "lower range" and the "upper range" of a CG, the problem of centrality can be stated in the following terms. Let there be, in the example given by Christaller, a series of CG's of diminishing range : 20 (maximal range), 19, 18, ..., 1 km (minimal range). This we may represent by writing :

$$CG_K - CG_{K-1} = CG_{K-1} - CG_{K-2} = \dots = CG_2 - CG_1 = 1.$$

Now let us suppose that the minimal différence in "range" of the CG is r kilometers.

$$CG_K - CG_{K-r} = CG_{K-r-1} = \dots = CG_{r+1} - CG_1 = r,$$

$$\text{where } K = 1, 2, 3, \dots, n \text{ and } r = 1, 2, \dots, n-1.$$

The problem of distribution of the CG stated by Christaller can now be generalized geometrically. Let there be two concentric circles C and C' of radii R and R' (R > R') and let K be the "ring" defined as being between the two circumferences. How can the ring K be covered with a minimum of circles C_i ?

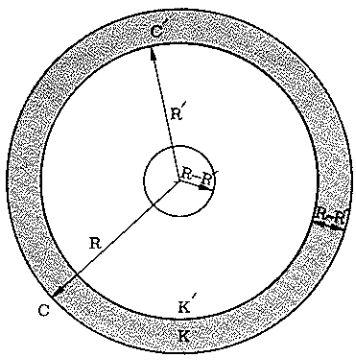

FIGURE 6

First it is demonstrated that it is impossible to cover the ring K with one or two circles.

Second it is demonstrated that the ring K can be covered by three unequal circles the centers of which are not necessarily equidistant from one another.

As a last step, when the width of the ring K increases (or, what amounts to the same thing, when the width of the ring K' diminishes), it is demonstrated that any geometrical figures whatsoever having 4, 5 or 6 sides also solve the problem of centrality posed by Christaller.

This results are summarized on the following figure 7.

Consequently, the geometrical solutions advocated by Christaller (construction of an equilateral triangle or of a regular hexagon) are valid only under the following conditions :

(1) All CG's have the same "range" : $R = R'$ (special case); the ring K does not exist; the CP's can be arranged around the initial CP at the summits of an equilateral triangle.

(2) The ratio of the "range" of the CG's is exactly $\frac{R'}{R} = \frac{\sqrt{3}}{2}$ (extreme case); the ring K exists and has a maximal width; the CP's must be arranged around the initial CP within the ring K' at the summits of an equilateral triangle.

(3) A CG has a "range" exactly equal to half the "range" of another CG : $R' = \frac{R}{}$ (special case); the ring K' does not exist; the CP's must be arranged around the initial CP at the summits of a regular hexagon.

(4) In general, the CG's have different ranges : $\frac{\sqrt{3}}{2} \leq \frac{R'}{R} < 1$; the ring K' exists; the CP's can be arranged around the initial CP at the summits of an equilateral triangle (special case) located within the ring K'.

4. Geographical Foundations for the Geometrical Solution of the Centrality Problem

The centrality problem, as it was stated by Christaller in 1933, is relevant and geometrically solvable if one avoids the fascination brought to bear by certain geometrical shapes like the equilateral triangle or the regular hexagon. In this way, unlike what Christaller thought, the problem of centrality has not one but an infinity of geometrical solutions. This apparently paradoxical result is due to the fact that the fundamental characteristics of the space of centrality are those not only of Euclidean but also of geographical space. This agrees with empirical observation, for if we join the nearest or most accessible inhabited places together with lines we obtain a lattice of triangles of any kind whatsoever. A second lattice, organized around the initial inhabited places, may be constructed on this first one. The new lattice is "central" in the sense in which we defined it at the beginning of section 2, on the basis of the problem stated by Christaller. The figures obtained are not regular but for the rarest exception, the probability of which, e.g. for the equilateral triangle, is practically nil. Finally, this result is far from trivial, since in an expanded and operational way it provides for the foundation of the centrality problem, the existence of which is obvious to all geographers even if its solution is not.

Furthermore, as the geometrical solution acknowledges, in a system of n central distances the K^{th} pair (of minimal and maximal distances) is related to the $n - 1$ other pairs of central distances. This theoretical property is verified by empirical studies of ranges of central goods in Europe as in the Americas. There are thus central distances for food products, luxury items, amusement, health services, schools, churches, prisons, etc. irregularly arranged and related to each other. Distance as defined by Christaller has all the properties, therefore, of Euclidean distance, and it has a further characteristic that makes a central distance of it.

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Euclidean distance and geographical distance are compatible but the latter cannot be reduced to the former. It is therefore possible to represent geographical distance graphically on a map (a Euclidean plane) by means of Euclidean distance without thereby reducing the geographical forms to their geometrical representation. For the geographer, the properties of geographical space are prior to those of Euclidean space. Geographical space is therefore not a "distortion" of geometrical space, as in the paradigm of centrality.

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